chang@umbc.edu

NP Completeness:

a machine model

$3COLOR \leq_m^p SAT$
$SAT \leq_m^p 3COLOR$

Recap: 3COLOR is equivalent to SAT.

sets.

<u>Defn</u>: Let A & B be decision problems.
We say A reduces to B (written $A \leq_m^p B$)
if there exists a polynomial-time computable function f s.t.

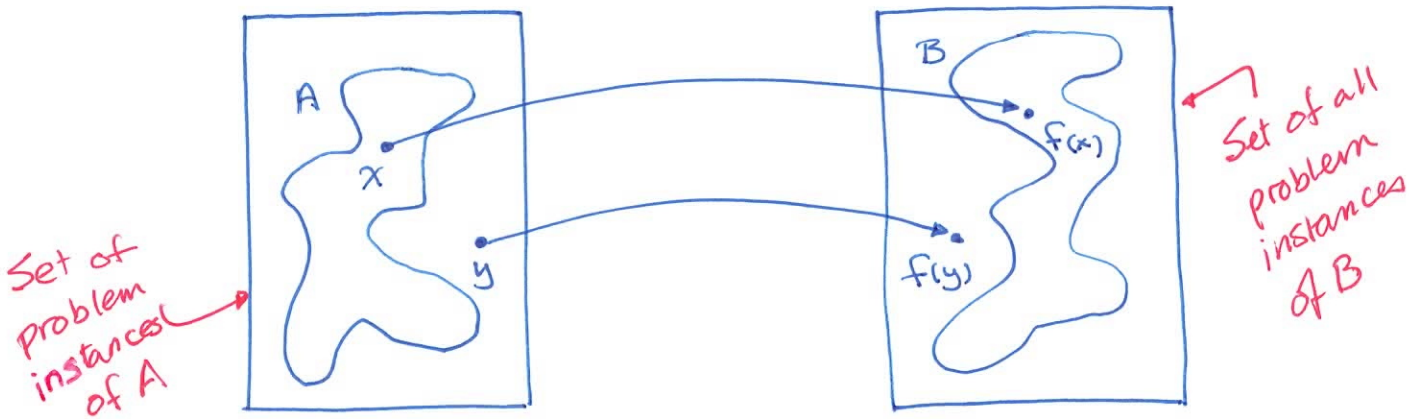
$$x \in A \iff f(x) \in B$$

"polynomial-time computable" = algorithm for f runs in $O(n^k)$ time for some $k \in \mathbb{N}$.

constant

$A \leq_m B$ + fast algorithm for B
$\Rightarrow$ "fast" algorithm for A

Suppose $A \le_m^P B$.



$$x \in A \Rightarrow f(x) \in B$$

$$y \notin A \Rightarrow f(y) \notin B$$

will define later

Problems equivalent to SAT & 3COLOR are NP-complete.

equivalence is transitive

$A \leq_m^p B \;\&\; B \leq_m^p C \Rightarrow A \leq_m^p C$

Thousands of problems are NP-complete.

Capture many important optimization problems, eg:

- Clique
- Vertex Cover
- Traveling Salesman Problem
- Partition
- 3D Matching.

will give you the flavor of a range of NP-complete problems.

Clique:

Input: undirected graph $G = (V, E)$
number k

as a subgraph

Question: Does G have a k-clique?

k-clique = k vertices in V s.t. any two vertices are connected by an edge.

3-Clique

4-Clique

5-Clique

Vertex Cover

Input: an undirected graph $G=(V,E)$
a number k

Question: does there exist a subset $V' \subseteq V$ s.t.
$|V'| \leq k$, and
for all edges $(u,v) \in E$, either $u \in V'$ or $v \in V'$?

similar to Hamiltonian Cycle

Traveling Salesman Problem

Input: an undirected graph $G=(V,E)$

a weight function $\omega: E \to \mathbb{R}^+$

a bound B

Question: Does there exist a simple cycle in G that visits every vertex exactly once s.t. the sum of the edge weights of the edges in the cycle is $\leq B$.

Partition

Input: n number $a_1, \ldots, a_n \in \mathbb{Z}^+$

Question: Does there exist a subset S of the numbers s.t.

$$\sum_{i \in S} a_i = \sum_{i \notin S} a_i$$

I.e., pick a subset of the numbers s.t. the sum of the numbers is exactly half of the total sum.

3-Dimensional Matching

Input: disjoint sets W, X, Y s.t. $n = |W| = |X| = |Y|$

$$M \subseteq W \times X \times Y$$

$\hookrightarrow M = \{ (w, x, y) \mid w, x \,\&\, y \text{ are "compatible"} \}$

Question: does there exist $M' \subseteq M$ s.t. $|M'| = n$

and no two elements of M' agree in any coordinate.

$$W = \{ w \mid \exists x \in X \,\&\, \exists y \in Y \quad (w, x, y) \in M' \}$$
$$X = \{ x \mid \exists w \in W \,\&\, \exists y \in Y \quad (w, x, y) \in M' \}$$
$$Y = \{ y \mid \exists w \in W \; \exists x \in X \quad (w, x, y) \in M' \}$$

each $w, x \,\&\, y$ appears exactly once

P = decision problems that can be solved by some algorithm that runs in time $O(n^k)$ for some constant k.

NP = decision problems that can be verified in P.

= guessing.

= "nondeterministic" polynomial time.

Check this is true for Clique, VC, 3DM, partition, etc.

P = NP means there is a free lunch.

Working definition of NP

A decision problem $A \in NP$ if

① $\exists\, B \in P$

② $\exists\, k \in \mathbb{N}$

$$x \in A \iff \exists y,\ |y| \le |x|^k,\ \text{s.t.}\ (x,y) \in B.$$

otherwise $NP = coNP$

Note: some properties are difficult to verify.

$\{(G,k) \mid$ the largest clique in G has $\le k$ vertices $\}$

$=$ G does not have cliques $> k$

Defn: A decision problem X is NP-complete, if

1. $X \in NP$
2. for all $Y \in NP$, $Y \leq_m^p X$

Cook's Theorem [1971] : SAT is NP-complete.

How to show that a new problem Q is NP-complete.

1. Show $Q \in NP$
2. Show $SAT \leq_m^p Q$

or some other known NP-complete problem.

Example: Vertex Cover (VC)

$VC = \{ (G,k) \mid \exists V' \subseteq V,\ |V'| \leq k$ and for all $(u,v) \in E$ either $u \in V'$ or $v \in V'$. $\}$

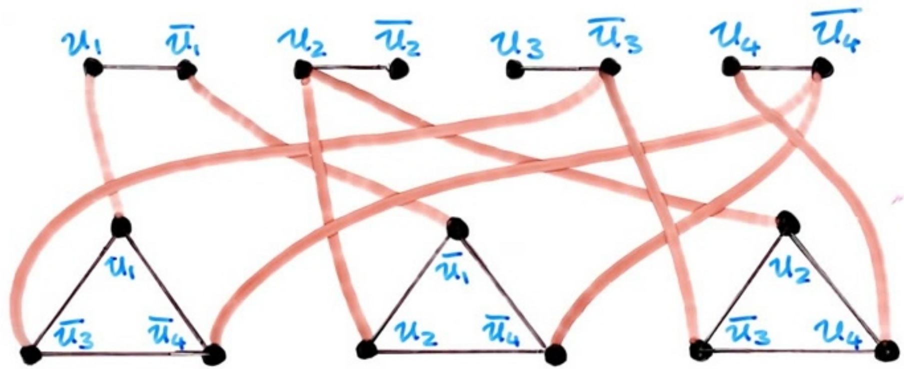
Show $VC \in NP$. Guess $V' \subseteq V$, check each edge in E.

much easier than showing $VC \leq_m^P E$.

3SAT $\leq^P_m$ VC

$$\phi = (u_1 \lor \overline{u_3} \lor \overline{u_4}) \land (\overline{u_1} \lor u_2 \lor \overline{u_4}) \land (u_2 \lor \overline{u_3} \lor u_4)$$

G:



ϕ has n variables & m clauses

G has $2n+3m$ nodes & $n+6m$ edges

$k = n + 2m$

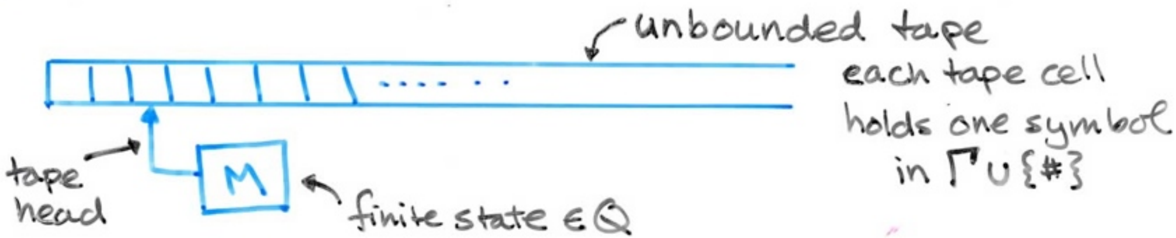
<u>Claim:</u>

$\phi \in$ 3SAT $\Leftrightarrow$ G has a vertex cover w/ k vertices

Cook's Theorem in 20 minutes

Hand Wavy Part

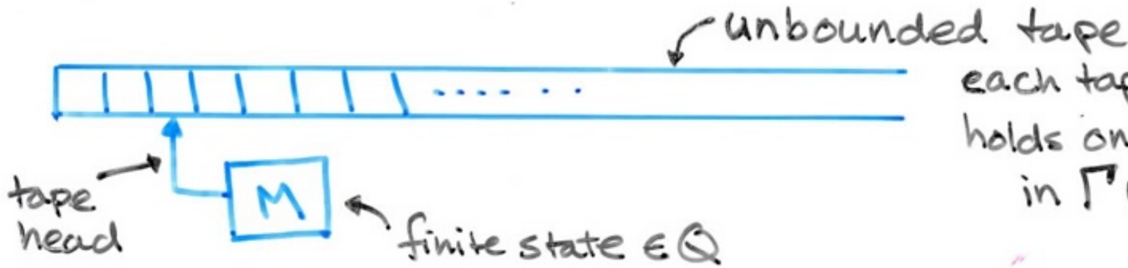
Turing machines



- In each step, a TM M can read one symbol of the tape under the tape head, enter a new state, replace the symbol underneath the tape head and move the tape head left or right.

skip

Turing machines



each tape cell holds one symbol in $\Gamma \cup \{\#\}$

- transition function

  $$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

- An input string x is accepted by a TM M if M starting in the start state & the tape head on the leftmost tape cell & x on the tape, enters a unique accepting state q_{acc} after a finite number of transitions.

- $x \in L(M)$ if x is accepted by TM M.

Church-Turing Thesis:

If A is a "computable" set, then $A = L(M)$ for some Turing machine M

Robustness of TM's

extra heads, tapes, ... do not add computational power to TM's.

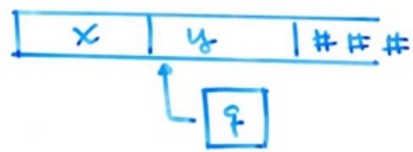
TM's & running time:

If $A \in P$ via the RAM model then $A = L(M)$ for some TM that makes a polynomial number of transitions.

Representing TM configurations

"instantaneous description" = ID

xqy means



Tape holds xy. Tape head reading first symbol of y.
Machine M in state q.

NOT SO HAND WAVY PART

Tableau: visual aid for thinking about
a sequence of ID's

Working defn of NP:

$A \in NP$ if $\exists\ B \in P$ and a polynomial $q(\ )$ s.t.

$x \in A \iff \exists\ y \in \Sigma^*$, $|y| \leq q(|x|)$ & $(x, y) \in B$.

Thus, $x \in A$ iff

$\exists$ a legal tableau

Starting with ID $q_0 x \# y$

s.t. M enters the accepting state q_{acc}

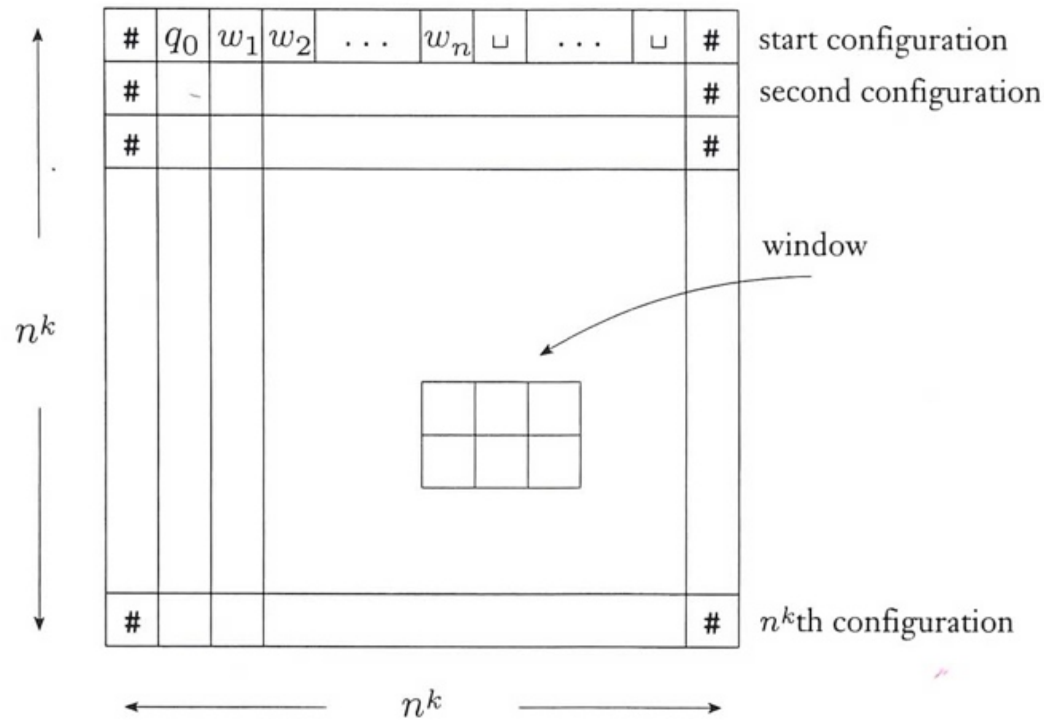


FIGURE 7.8
A tableau is an $n^k \times n^k$ table of configurations

[Sipser]

NOT HAND WAVY PART

Use Boolean formulas to check a tableau is legal

Each cell can hold a state, a tape symbol, or a #. Let $C = Q \cup \Gamma \cup \{\#\}$.

Each cell indexed by i, j, $1 \le i \le n^k$ & $1 \le j \le n^k$.

For each cell i, j and each symbol $s \in C$

$x_{i,j,s}$ is true "means" cell i, j holds symbol s.

Enforce that each cell has a symbol

$$\phi_{cell1} = \bigwedge_{i,j} \bigvee_{s \in C} x_{i,j,s}$$

Enforce that each cell has no more than one symbol

$$\phi_{cell2} = \bigwedge_{i,j} \bigwedge_{\substack{s,t \in C \\ s \neq t}} \left(\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}} \right)$$

Enforce that initial configuration is $q_0 x \# y$

$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \wedge x_{1,3,-} , \wedge x_{1,4,-}$$

$$\wedge\ x_{1,n+2,\#} \wedge x_{1,m+2,\#} \wedge \cdots\cdot \wedge x_{1,n^k,\#}$$

where $n = |x|$, $m = |x \# y|$

This ensures the first line of the tableau is

$$\# \, q_0 \, x \, \# \, y \, \# \# \# \cdots \#$$

for <u>some</u> y.

Enforce that M entered the accepting state

$$\phi_{acc} = \bigvee_{i,j} x_{i,j,q_{acc}}$$

Enforce that line i+1 of the tableau follows from line i.

Observation: Only need to check that all 2×3 "windows" are legal.

LEGAL = set of all legal 2×3 windows $\subseteq C \times C \times C \times C \times C \times C$

Note: |LEGAL| is finite & constant.

$$\phi_{move} = \bigwedge_{i,j} \text{2×3 window at index } i,j \text{ is LEGAL}$$

$$= \bigwedge_{i,j} \bigwedge_{\substack{(a_1 \cdots a_6) \\ \notin \text{LEGAL}}} \left(\overline{x_{i-1,j,a_1}} \vee \overline{x_{i,j,a_2}} \vee \overline{x_{i+1,j,a_3}} \vee \overline{x_{i-1,j+1,a_4}} \vee \overline{x_{i,j+1,a_5}} \vee \overline{x_{i+1,j+1,a_6}} \right)$$

i.e., a legal window a i,j has one entry different from every illegal window.

(a)

a	q_1	b
q_2	a	c

(b)

a	q_1	b
a	a	q_2

(c)

a	a	q_1
a	a	b

(d)

#	b	a
#	b	a

(e)

a	b	a
a	b	q_2

(f)

b	b	b
c	b	b

FIGURE 7.9
Examples of legal windows

(a)

a	b	a
a	a	a

(b)

a	q_1	b
q_1	a	a

(c)

b	q_1	b
q_2	b	q_2

FIGURE 7.10
Examples of illegal windows

[Sipser]

Claim: $x \in A$

$\Longleftrightarrow$ $\exists$ a legal tableau starting with $q_0 x \# y$

$\Longleftrightarrow$ $\phi_{cell1} \wedge \phi_{cell2} \wedge \phi_{start} \wedge \phi_{acc} \wedge \phi_{move} = \phi$ is satisfiable.

Note: ϕ is in conjunctive normal form